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Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 68 (2004) 839–846

EXTENSION OF THE DOMAIN OF APPLICABILITY OF THE INTEGRAL STABILITY CRITERION (EXTREMUM PROPERTY) IN SYNCHRONIZATION PROBLEMS†

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(*Received* 13 *October* 2003)

The method of direct separation of motions is used to justify an extended formulation of the integral stability criterion which enables both "simple" and "non-simple" cases of problems of the synchronization of objects with almost uniform rotations to be considered. It is shown, by means of examples, that the results found previously by Poincaré's small parameter method and by direct separation of motions, which require laborious computations, can be obtained much more simply by using an extended formulation of the integral stability criterion.

The investigation of synchronization can be simplified considerably, and the results given a more convenient form, if the so-called integral stability criterion for synchronous motions holds [1–3]. In multiple synchronization problems, however, and in many problems involving simple synchronization of identical unbalanced vibrodrivers and a fairly wide class of cases of practical importance (those called "non-simple"), the integral criterion, in the form obtained by the Poincaré–Lyapunov small parameter method, does not allow the values of the phase rotations of the vibrodriver rotors in stable synchronized motions to be determined [2–6].

On the basis of the Poincaré and Lyapunov methods, the following remarkable property has been observed in the synchronous motions of objects with almost uniform rotations and several other dynamical objects [1–3]: stable synchronous motions correspond to points of a strict coarse minimum of a certain function D (the "potential function") of the so-called generating parameters – the initial rotation phases $\alpha_1, \ldots, \alpha_k$ (in the self-synchronization problem – the phase differences $\alpha_s - \alpha_k$, where k is the number of rotations; see below). In cases of importance for applications, the potential function D is minus the mean Lagrangian of the system over the period of rotations; in other, slightly more special cases – it is the mean Lagrangian of the oscillatory part of the system, that is, the system with "stopped" rotations.

It has been proved by the use of the integral criterion, under fairly general assumptions, that a wide class of objects display a tendency to synchronization, and various important applied problems have been solved [2]. The extremum property of synchronous ("resonant") motions has also been established for motions of celestial bodies (see, e.g. [2, 7–9]).

At the same time, there are cases in which the integral criterion in the form described does not enable one to find phase values in stable synchronous motions. This is the case, in particular, in problems of multiple synchronization for vibrodrivers in quasi-linear systems and in various synchronization problems for several (more than three) identical vibrodrivers [2, 4–6]. In these cases, known as "non-simple", the function D does not depend on some of the phases, and its minimum is therefore not strict.

It will be shown below that the integral criterion remains valid if the function D is calculated not on the basis of the generating solution but more precisely – to the extent that this is necessary to establish its strict minimum.

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 938–947, 2004.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.11.005

1. THE PROBLEM OF THE SYNCHRONIZATION OF OBJECTS WITH ALMOST UNIFORM ROTATIONS

This problem may be formulated as follows [2, 3]. Consider a system with generalized coordinates φ_s ($s = 1, \ldots, k$) ("rotational coordinates") and u_r ($r = 1, \ldots, \nu$) ("oscillatory coordinates"). It is assumed here that the Lagrangian of the system can be expressed as

$$L = \frac{1}{2}\sum_{s=1}^{k} I_s\dot{\varphi}_s^2 + L_{\sim}(\varphi, \dot{\varphi}, u, \dot{u}, \omega t) \tag{1.1}$$

and the non-conservative generalized forces corresponding to the rotational coordinates as

$$Q_{\varphi_s} = -k_s(\dot{\varphi}_s - \sigma_s n_s\omega) + \sigma_s k_s(\omega_s - n_s\omega) \tag{1.2}$$

where I_s, k_s and ω are positive constants; $\sigma_s = \pm 1$, n_s are positive integers, and ω_s are the so-called partial angular velocities of rotation – the angular velocities of the rotations in the case that there are no oscillatory motions (u_r = const).

In the case of the synchronization problem for rotors, the difference between the torque $L_s(\dot{\varphi}_s)$ turning the sth rotor and the torque of the resistance forces $R_s(\dot{\varphi}_s)$ can be expressed in the form (1.2). It is assumed that the equations of motion of the system may be written as

$$I_s\ddot{\varphi}_s + k_s(\dot{\varphi}_s - \sigma_s n_s\omega) = \mu\Phi_s, \quad s = 1, \ldots, k \tag{1.3}$$

$$E_{u_r}(L_{\sim}) = Q_{u_r}, \quad r = 1, \ldots, \nu \tag{1.4}$$

where $E_q = \frac{d}{dt}\frac{\partial}{\partial\dot{q}} - \frac{\partial}{\partial q}$ is the Euler operator and Q_q is the non-conservative generalized force corresponding to the coordinate q,

$$\mu\Phi_s = \sigma_s k_s(\omega_s - n_s\omega) - E_{\varphi_s}(L_{\sim}) \tag{1.5}$$

$\mu > 0$ being a small parameter. The functions $L_{\sim}$ and Q_{u_r} may depend both on the generalized coordinates and velocities and on the time $\tau = \omega t$, being 2π-periodic functions of φ_s and τ; the functions $L_{\sim}$ and Q_{u_r} may depend on μ. The functions are assumed to be sufficiently smooth to guarantee the existence of all solutions and expansions considered below.

The generating equations ($\mu = 0$) corresponding to Eqs (1.3) and (1.4) admit of a family of solutions

$$\varphi_s^0 = \sigma_s(n_s\omega t + \alpha_s) \tag{1.6}$$

corresponding to uniform rotations at velocities $|\dot{\varphi}_s^0| = n_s\omega$ and certain arbitrary phases α_s.

The synchronization problem is to find conditions for the existence and stability of solutions of Eqs (1.3) and (1.4) of the form

$$\varphi_s = \sigma_s[n_s\omega t + \alpha_s + \mu\psi_s^{(p)}(\omega t, \mu)], \quad u_r = u_r^{(p)}(\omega t, \mu) \tag{1.7}$$

where $\psi_s^{(p)}$ and $u_r^{(p)}$ are 2π-periodic functions of $\tau = \omega t$. A solution of this problem by Poincaré–Lyapunov small parameter methods is presented, in particular, in [2].

2. SOLUTION OF THE PROBLEM BY DIRECT SEPARATION OF MOTIONS

To solve the problem by the method of direct separation of motions [3], a solution of the equations is sought in the form

$$\varphi_s = \sigma_s[n_s\omega t + \alpha_s(t) + \psi_s(t, \omega t, \mu)], \quad u_r = u_r(t, \omega t, \mu) \tag{2.1}$$

where $\alpha_s(t)$ are "slow", and ψ_s and u_r are "fast" 2π-periodic components (t is the "fast" and $\tau = \omega t$ is the "slow" time variable, and ω is a "large" parameter), and it is assumed that

$$\langle\psi_s(t, \omega t, \mu)\rangle = 0, \quad \langle u_r(t, \omega t, \mu)\rangle = 0 \tag{2.2}$$

Finally, since by the representations (2.1) the functions $\dot{\alpha}_s$ occur in all relations only in the combination $n_s\omega + \dot{\alpha}_s$, it follows by (2.3) that the asterisk in equalities (2.9), (2.10) and (2.12) may be omitted. As a result, the equations of slow motions (2.4) may be written in the form

$$I_s\ddot{\alpha}_s + k_s\dot{\alpha}_s = -\frac{\partial D}{\partial \alpha_s}, \quad s = 1, \dots, k \tag{2.13}$$

where

$$D = -(\Lambda + B), \quad \frac{\partial B}{\partial \alpha_s} = k_s(\omega_s - n_s\omega) + \sum_{r=1}^{\nu}\left\langle [Q_{u_r}]\frac{\partial u_r}{\partial \alpha_s}\right\rangle \tag{2.14}$$

D is the potential function and B is the so-called potential of mean non-conservative generalized forces, corresponding to the oscillatory coordinates (it is assumed that such a potential exists).

3. EXTENDED FORMULATION OF THE INTEGRAL CRITERION

Under the assumptions adopted above, Eq. (2.13) immediately implies (see, e.g. [10]) the validity of a generalized formulation of the integral criterion (the extremum property) for synchronous motions of objects with almost uniform rotations (in the synchronization problem one considers asymptotic stability in the small, and in self-synchronization problems – asymptotic orbital stability [1–3]). Stable synchronous motions correspond to values of the phase $\alpha_s = \text{const}$, which determine strict coarse minima of the potential function $D = D(\alpha_1, \dots, \alpha_k)$ (in the self-synchronization problem: $D = D(\alpha_1 - \alpha_k, \dots, \alpha_{k-1} - \alpha_k)$ and one considers minima of this function as a function of the phase differences $\alpha_s - \alpha_k$); unlike the previous formulation, the function D may be evaluated not necessarily in the generating approximation ($\mu = 0$) but to within any approximation with respect to μ, with two reservations:

(1) it is assumed in addition that α_s varies slowly compared with ψ_s and that $\dot{\alpha}_s$ is small compared with $n_s\omega$, that is, $\dot{\alpha}_s \ll \dot{\psi}_s$ and $\dot{\alpha}_s \ll n_s\omega$

(2) expressions (2.10) and (2.11) must be small to a higher order than the terms included in the evaluation of the functions Λ and B.

The more accurate evaluation of the function D makes it possible to establish the existence of strict minima in those ("non-simple") cases in which this function, in principle, has no such minima in the generating approximation.

As before (see [1–3]), the expression for D may be simplified considerably under certain additional assumptions.

1. If the partial velocities ω_s are appropriate multiples of the synchronous velocity ω, that is, $\omega_s = n_s\omega$, and the non-conservative forces Q_{u_r} are negligibly small, then

$$D = -\Lambda = -\langle [L]\rangle \tag{3.1}$$

that is, the potential function is the Lagrangian of the system averaged over a period, evaluated for the function (1.7).

2. Suppose the system is linear in the oscillatory coordinates and the function L can be written as

$$L = L^* + L^{(\mathrm{I})} + L^{(\mathrm{II})} \tag{3.2}$$

with

$$L^* = \sum_{s=1}^{k} L_s(\dot{\varphi}_s, \varphi_s) + \sum_{r=1}^{\nu} f_r(\dot{\varphi}_1, \dots, \dot{\varphi}_k, \varphi_1, \dots, \varphi_k)\dot{u}_r + \sum_{s=1}^{k} F_s(\varphi_s) \tag{3.3}$$

$$L^{(\mathrm{I})} = \frac{1}{2}\sum_{r=1}^{\nu}\sum_{j=1}^{\nu}(a_{rj}\dot{u}_r\dot{u}_j - b_{rj}u_ru_j), \quad L^{(\mathrm{II})} = \Psi(\dot{\varphi}_1, \dots, \dot{\varphi}_k, \varphi_1, \dots, \varphi_k) \tag{3.4}$$

where a_{rj} and b_{rj} are constants, and L_s, f_r, F_s and Ψ are functions of the variables indicated; moreover, L_s, f_r and F_s are periodic in φ_s with period 2π. Suppose in addition that the generalized forces Q_{u_r} are either absent or negligibly small.

Then the following relations hold

$$\frac{\partial \Lambda}{\partial \alpha_s} = \frac{\partial(\Lambda^{(\mathrm{II})} - \Lambda^{(\mathrm{I})})}{\partial \alpha_s} + O(\mu)$$
$$(\Lambda^{(\mathrm{I})} = \langle [L^{(\mathrm{I})}]\rangle, \quad \Lambda^{(\mathrm{II})} = \langle [L^{(\mathrm{II})}]\rangle) \tag{3.5}$$

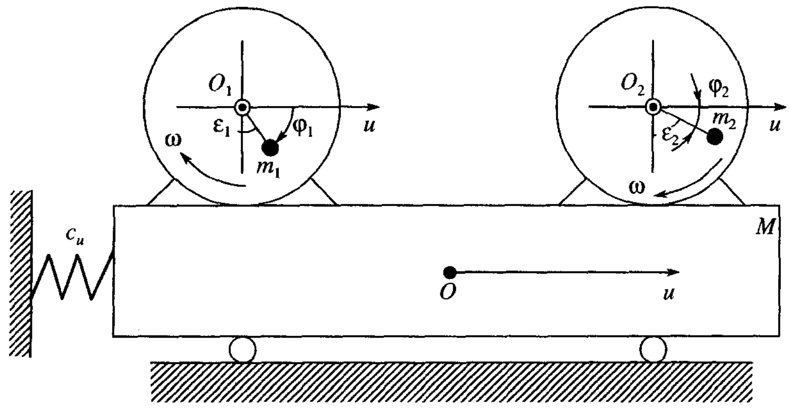

Fig. 1

These relations differ from those which hold for the generating solution (see [3, Chap. 3, Eq. (2.20)]) in that they hold to within terms of the order of μ; this is easily established by calculations analogous to those presented in [3].

As a result, in this case, on the assumption that terms of the order of μ may be neglected in (3.5), we can take

$$D = \Lambda^{(I)} - \Lambda^{(II)} \tag{3.6}$$

and, in the case when $\Lambda^{(II)} = 0$, put

$$D = \Lambda^{(I)} \tag{3.7}$$

In this case, therefore, the potential function is the Lagrangian of only the oscillatory part of the system, averaged over a period.

Incidentally, the same result may be obtained using the Hamilton variational principle, modifying and extending Lur'ye's work in [11].

4. EXAMPLES COMPARED WITH RESULTS OBTAINED BY CLASSICAL METHODS

4.1. *Double synchronization of two vibrodrivers on a vibrating platform (a supporting body) with one degree of freedom* (Fig. 1)

Two unbalanced vibrodrivers mounted on a supporting body are set in rotation by electric induction motors. The supporting body can move relative to a stationary base in a certain fixed direction Ou and is connected to the base by linear elastic elements. The equations of motion (1.3) and (1.4) for this problem may be written as follows [2]:

$$\begin{gathered} I_s\ddot{\varphi}_s + k_s(\dot{\varphi}_s - \sigma_s n_s\omega) = \mu[L_s(\sigma_s n_s\omega) - R(n_s\omega)] + m_s\varepsilon_s(\ddot{u}\sin\varphi_s + g\cos\varphi_s)], \\ s = 1, 2; \quad n_1 = 1, \quad n_2 = 2 \end{gathered} \tag{4.1}$$

$$M\ddot{u} + c_u u = \sum_{j=1}^{2} m_j\varepsilon_j(\ddot{\varphi}_j\sin\varphi_j + \dot{\varphi}_j^2\cos\varphi_j) \tag{4.2}$$

where m_s is the mass of the rotor of the sth vibrodriver, ε_s is its eccentricity, I_s is its moment of inertia about the axis of rotation, $k_s > 0$ are constant coefficients characterizing the damping, M is the mass of the supporting body, allowing for the mass of the vibrodrivers, c_u is the stiffness of the elastic elements, and g is the acceleration due to gravity.

When the problem is solved by direct separation of the motions, Eqs (4.1) and (4.2) are replaced, via representations (2.1), by the equations of the slow and fast motions (2.4) and (2.5), where

$$\Phi_s = L_s(n_s\omega) - R_s(n_s\omega) + m_s\varepsilon_s[\ddot{u}\sin(n_s\omega t + \alpha_s + \psi_s) + g\cos(n_s\omega t + \alpha_s + \psi_s)]$$

Equation (2.6), in turn, becomes

$$M\ddot{u} + c_u u = \sum_{j=1}^{2} m_j\varepsilon_j[\ddot{\psi}_j\sin(n_j\omega t + \alpha_j + \psi_j) + (n_j\omega + \dot{\psi}_j)^2\cos(n_j\omega t + \alpha_j + \psi_j)] \tag{4.3}$$

The periodic solutions of the equations of fast motions (2.5) are expanded in series in powers of μ. Note that in the case under consideration – multiple synchronization – it is no longer sufficient to take $\psi_s = 0$ in the first approximation. At the same time, as will be shown below, it will suffice to evaluate ψ_s to within terms of at most the first order. On that assumption we find ($p^2 = c_u/M$)

$$\mu\psi_s = -\frac{(sm_s\varepsilon_s\omega)^2}{8MI_s[(s\omega)^2 - p^2]}\sin 2(s\omega t + \alpha_s) - \frac{m_s\varepsilon_s g}{(s\omega)^2 I_s}\cos(s\omega t + \alpha_s) +$$

$$+\frac{1}{2}(-1)^s(31 - 15s)\frac{m_1\varepsilon_1 m_2\varepsilon_2\omega^2}{MI_s[(2\omega/s)^2 - p^2]}\left[\frac{1}{9}\sin(3\omega t + \alpha_1 + \alpha_2) + (-1)^s\sin(\omega t + \alpha_2 - \alpha_1)\right]$$

$s = 1, 2$

Then the solution of Eq. (4.3) takes the form

$$u = -\frac{m_1\varepsilon_1\omega^2}{M(\omega^2 - p^2)}\cos(\omega t + \alpha_1) - $$
$$-\frac{4m_2\varepsilon_2\omega^2}{M(4\omega^2 - p^2)}\cos(2\omega t + \alpha_2) - \frac{2m_1^2\varepsilon_1^2 g\omega^2}{MI_1\omega^2(4\omega^2 - p^2)}\sin 2(\omega t + \alpha_1) \tag{4.4}$$

where we have written out only those terms that affect the computation of the mean Lagrangian to within the approximation considered.

Substituting expression (4.4) into the expressions for the kinetic and potential energy of the oscillatory part of the system

$$T^{(1)} = \frac{1}{2}M\dot{u}^2, \quad \Pi^{(1)} = \frac{1}{2}c_u u^2$$

and averaging them over one period, we obtain the potential function for the case in question

$$D = \Lambda^{(1)} = \langle[L^{(1)}]\rangle = \langle[T^{(1)} - \Pi^{(1)}]\rangle = \frac{4m_1^2\varepsilon_1^2 m_2\varepsilon_2 g\omega^2}{MI_1(4\omega^2 - p^2)}\sin(2\alpha_1 - \alpha_2)$$

As a result, the equations of slow motion, which also describe the motion in the neighbourhood of steady synchronous motions α_s = const, may be represented in the form (2.13).

The expression obtained for the function D yields expressions for the vibrational moments

$$W_s = \frac{\partial\Lambda^{(1)}}{\partial\alpha_s} = \frac{4(5 - 3s)m_1^2\varepsilon_1^2 m_2\varepsilon_2 g\omega^2}{MI_1(4\omega^2 - p^2)}\cos(2\alpha_1 - \alpha_2), \quad s = 1, 2 \tag{4.5}$$

These expressions are exactly the same as those obtained by a more complicated procedure using Poincaré's method [4]. Consequently, all the other results are identical as well.

4.2. *Double synchronization of three unbalanced vibrodrivers mounted symmetrically on a softly vibration-insulated plane-oscillating rigid body* (Fig. 2)

The equations of motion of the system are

$$I_s\ddot{\varphi}_s + k_s(\dot{\varphi}_s - \sigma_s n_s\omega) = \mu[L_s(\sigma_s n_s\omega) - R_s(n_s\omega) +$$
$$+ m_s\varepsilon_s(\ddot{x}\sin\varphi_s + \ddot{y}\cos\varphi_s - r_s\ddot{\phi}\cos\varphi_s + g\cos\varphi_s)], \quad s = 1, 2, 3$$

$$M\ddot{x} = \sum_{j=1}^{3} m_j\varepsilon_j(\ddot{\varphi}_j\sin\varphi_j + \dot{\varphi}_j^2\cos\varphi_j)$$

$$M\ddot{y} = \sum_{j=1}^{3} m_j\varepsilon_j(\ddot{\varphi}_j\cos\varphi_j - \dot{\varphi}_j^2\sin\varphi_j)$$

$$I\ddot{\phi} = \sum_{j=1}^{3} m_j\varepsilon_j r_j(\dot{\varphi}_j^2\sin\varphi_j - \ddot{\varphi}_j\cos\varphi_j)$$

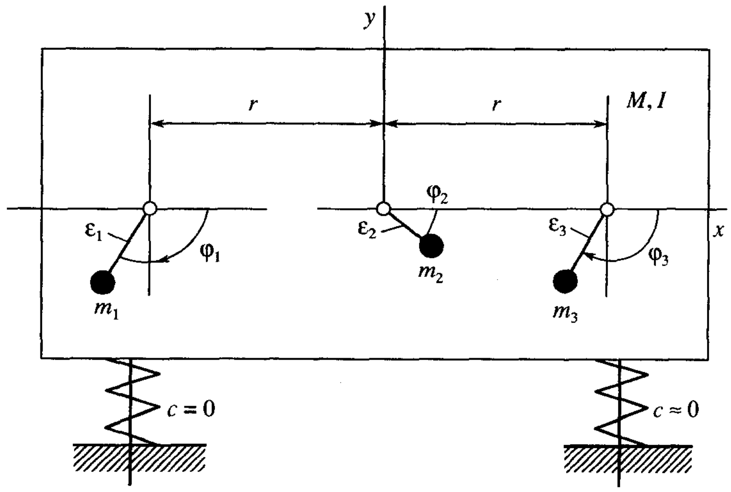

Fig. 2

where I is the moment of inertia of the supporting body and r_s is the distance from the axis of the sth vibrodriver to the centre of gravity of the supporting body. The outer vibrodrivers are identical and their axes are equidistant from the centre of gravity, that is, $m_1\varepsilon_1 = m_3\varepsilon_3$, $I_1 = I_3$, $r_1 = r_3 = r$, $r_2 = 0$. All the vibrodrivers rotate in the same direction, the frequency of the central vibrodriver being twice that of the outer ones: $n_1 = n_3 = 1$, $n_2 = 2$.

Applying the method of direct separation of motions via the representations (2.1), one readily arrives at the equations of the slow and fast motion (2.4)–(2.6), where, in this case,

$$\Phi_s = L_s(n_s\omega) - R_s(n_s\omega) + m_s\varepsilon_s[\ddot{x}\sin\chi_s + \ddot{x}\psi_s\cos\chi_s +$$
$$+ \ddot{y}\cos\chi_s - \ddot{y}\psi_s\sin\chi_s + r_s\ddot{\varphi}\cos\chi_s - r_s\ddot{\varphi}\psi_s\sin\chi_s + g\cos\chi_s - g\psi_s\sin\chi_s], \quad \chi_s = n_s\omega t + \alpha_s$$

First expanding the periodic solutions of the equations of fast motion (2.4) (in the same approximation as in the previous example), we find

$$\mu\psi_s = \frac{4m_1\varepsilon_1 m_2\varepsilon_2}{MI_1}\sin(\omega t + \alpha_2 - \alpha_s) - \frac{m_1\varepsilon_1 g}{I_1\omega^2}\cos(\omega t + \alpha_s) +$$
$$+ \frac{(m_1\varepsilon_1 r)^2}{8I_1}[\sin 2(\omega t + \alpha_s) - \sin(2\omega t + \alpha_1 + \alpha_3)], \quad s = 1, 3$$

$$\mu\psi_2 = -\frac{m_1\varepsilon_1 m_2\varepsilon_2}{MI_2}[\sin(\omega t + \alpha_2 - \alpha_1) + \sin(\omega t + \alpha_2 - \alpha_3)] - \frac{m_2\varepsilon_2 g}{4I_2\omega^2}\cos(2\omega t + \alpha_2)$$

Then Eqs (2.6) become

$$\ddot{x} = \frac{m_1\varepsilon_1\omega^2}{M}[\cos(\omega t + \alpha_1) + \cos(\omega t + \alpha_3)] + \frac{4m_2\varepsilon_2\omega^2}{M}\cos(2\omega t + \alpha_2) +$$
$$+ \frac{2m_1^2\varepsilon_1^2 g}{MI_1}[\sin 2(\omega t + \alpha_1) + \sin 2(\omega t + \alpha_3)]$$

$$\ddot{y} = -\frac{m_1\varepsilon_1\omega^2}{M}[\sin(\omega t + \alpha_1) + \sin(\omega t + \alpha_3)] - \frac{4m_2\varepsilon_2\omega^2}{M}\sin(2\omega t + \alpha_2) + \quad (4.6)$$
$$+ \frac{2m_1^2\varepsilon_1^2 g}{MI_1}[\cos 2(\omega t + \alpha_1) + \cos 2(\omega t + \alpha_3)]$$

$$\ddot{\varphi} = -\frac{m_1\varepsilon_1 r\omega^2}{I}[\sin(\omega t + \alpha_1) - \sin(\omega t + \alpha_3)]$$

where we have written out only those terms that affect the computation of the Lagrangian to within the approximation considered.

In view of the assumption concerning the softness of the elastic supports, the mean value of the Lagrangian is simply the mean kinetic energy of the supporting body

$$D = \Lambda^{(1)} = \langle[T^{(1)}]\rangle = \frac{1}{2}\langle[M(\dot{x}^2+\dot{y}^2)+I\dot{\varphi}^2]\rangle =$$

$$= \left(\frac{m_1^2\varepsilon_1^2\omega^2}{M} - \frac{m_1^2\varepsilon_1^2 r^2\omega^2}{2I}\right)\cos(\alpha_1-\alpha_3) + \frac{2m_1^2\varepsilon_1^2 m_2\varepsilon_2 g}{I_1 M}[\sin(2\alpha_1-\alpha_2)+\sin(2\alpha_3-\alpha_2)]$$

Finally, taking the equations for the slow components (mean phases of rotation of the rotors α_s) into consideration, we obtain the system

$$\begin{aligned}
&I_s\ddot{\alpha}_s + k_s\dot{\alpha}_s = L_s(\omega) - R_s(\omega) + \\
&+(2-s)\frac{m_1^2\varepsilon_1^2\omega^2}{M}\left(1-\frac{Mr^2}{2I}\right)\sin(\alpha_1-\alpha_3) - \frac{4m_1^2\varepsilon_1^2 m_2\varepsilon_2 g}{I_1 M}\cos(2\alpha_s-\alpha_2), \quad s = 1, 3 \\
&I_2\ddot{\alpha}_2 + k_2\dot{\alpha}_s = L_2(2\omega) - R_2(2\omega) + \\
&+\frac{2m_1^2\varepsilon_1^2 m_2\varepsilon_2 g}{I_1 M}[\cos(2\alpha_1-\alpha_2)+\cos(2\alpha_3-\alpha_2)]
\end{aligned} \tag{4.7}$$

Equations (4.7) of the slow processes leading to steady synchronous motion are exactly the same as those obtained by the more complicated method (using the method of direct separation of motions [5]). Hence all the other results are also the same. It should be emphasized that in the double synchronization problems considered here, it proved sufficient to solve Eqs (2.5) to within terms of the order of μ. Equations (2.6) are solved in the same way as in the "sample" case, but taking into account the greater accuracy in the solution of Eqs (2.5).

This research was supported financially by the Russian Foundation for Basic Research (03-01-00621) and Programme 19 of the Presidium of the Russian Academy of Sciences.

REFERENCES

1. BLEKHMAN, I. I., Justification of the integral criterion of stability of motion in problems on the self-synchronization of vibrators. *Prikl. Mat. Mekh.*, 1960, **24**, 6, 1100–1103.
2. BLEKHMAN, I. I., *Synchronization in Science and Technology*. ASME Press, New York, 1981.
3. BLEKHMAN, I. I., *Vibrational Mechanics. Nonlinear Dynamic Effects. General Approach. Applications*. World Scientific, Singapore, 2000.
4. BARZUKOV, O. P., Multiple synchronization in a system of weakly-coupled objects with one degree of freedom. *Prikl. Mat. Mekh.*, 1972, **36**, 2, 225–231.
5. YAROSHEVICH, H. P., The use of the self-synchronization effect in the excitation of biharmonic vibrations. *Problemy Mashinostroyeniya i Nadezhnosti Mashin*, 1990, 6, 23–27.
6. MALAKHOVA, O. Z., A singular case in the theory of the self-synchronization of mechanical vibrodrivers. Izv. Akad. Nauk SSSR. MTT, 1990, 1, 29–36.
7. OVENDEN, M. W., FEAGIN, T. and GRAFF, O., On the principle of least interaction action and the Laplacean satellites of Jupiter and Uranus. *Celest. Mech.*, 1974, **8**, 4, 455–471.
8. BELETSKII, V. V., Extremum properties of resonant motions. In *Stability of Motion. Analytical Mechanics. Control of Motion*. Nauka, Moscow, 1981, 41–55.
9. KHENTOV, A., On the principle of strong interaction for the resonant orbital motions of some celestial bodies. *Mathematics and Computers Simulation*, 2000, **58**, 4–6, 423–434.
10. MERKIN, D. R., *Introduction to the Theory of Stability of Motion*. Nauka, Moscow, 1987.
11. LUR'YE, A. I., Some self-synchronization problems. In *Proc. International Conference on Non-linear Oscillations*, Vol. 3. Inst. Mat. Akad. Nauk UkrSSR, Kiev, 1970, 449–455.

Translated by D.L.